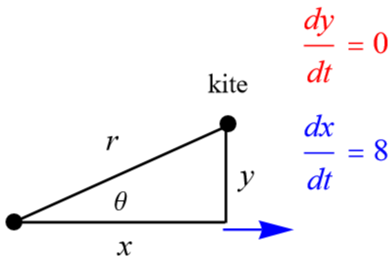
Exercise 30

A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

Solution

At any time, the kite's string is the hypotenuse of a right triangle.



The Pythagorean theorem holds for any right triangle.

$$r^2 = x^2 + y^2$$

Because we want to know $d\theta/dt$ when $r = 200$, start with a trigonometric function, which relates θ to the sides of the triangle.

$$\sin\theta = \frac{y}{r}$$

Write it in terms of x and y, since dx/dt and dy/dt are known.

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Take the derivative of both sides by using the chain and quotient rules.

$$\frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{y}{\sqrt{x^2 + y^2}}\right)$$

$$(\cos\theta)\cdot\frac{d\theta}{dt} = \frac{\left[\frac{d}{dt}(y)\right]\sqrt{x^2+y^2} - \left[\frac{d}{dt}\left(\sqrt{x^2+y^2}\right)\right]y}{x^2+y^2}$$

$$\left(\frac{x}{r}\right)\frac{d\theta}{dt} = \frac{\left(\frac{dy}{dt}\right)\sqrt{x^2+y^2} - \left[\frac{1}{2}(x^2+y^2)^{-1/2}\cdot\frac{d}{dt}(x^2+y^2)\right]y}{r^2}$$

$$= \frac{(0)\sqrt{x^2+y^2} - \left[\frac{1}{2\sqrt{x^2+y^2}}\cdot\left(2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt}\right)\right]y}{r^2}$$

Continue the simplification.

$$\left(\frac{x}{r}\right)\frac{d\theta}{dt} = \frac{-\frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)y}{r^2}$$

$$= \frac{-\frac{1}{r}[x(8) + y(0)]y}{r^2}$$

$$= -\frac{8xy}{r^3}$$

Solve for $d\theta/dt$.

$$\frac{d\theta}{dt} = -\frac{8y}{r^2}$$

Therefore, the rate of change of the angle between the string and the horizontal when the string is 200 feet long is

$$\left.\frac{d\theta}{dt}\right|_{r=200} = -\frac{8(100)}{(200)^2} = -\frac{1}{50}\ \frac{\text{radians}}{\text{second}} = -0.02\ \frac{\text{radians}}{\text{second}}.$$